

COUNTING ALL CUBES IN $\{0, 1, \dots, n\}^3$

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ABSTRACT. In this paper we describe a procedure of calculating the number cubes that have coordinates in the set $\{0, 1, \dots, n\}$. We adapt the code that appeared in [11] developed to calculate the number of regular tetrahedra with coordinates in the set $\{0, 1, \dots, n\}$. The idea is based on the theoretical results obtained in [13]. We extend then the sequence A098928 in the Online Encyclopedia of Integer Sequences [16] to the first one hundred terms.

1. INTRODUCTION

In this paper we continue and, at the same time revise, some of the work begun in the sequence of papers [2], [9]-[13] about equilateral triangles, regular tetrahedra, and other regular polyhedra, all having integer coordinates. Very often we will refer to this property by saying that the various objects are in \mathbb{Z}^3 . Strictly speaking these geometric objects are defined as being more than the set of their vertices that determines them, but for us here, these are just the vertices. So, for instance, an equilateral triangle is going to be a set of three points in \mathbb{Z}^3 for which the Euclidean distances between every two of these points are the same. The main purpose of the paper is to take a close look at the cubes in \mathbb{Z}^3 . One can imagine easily such cubes by taking the faces parallel to the planes of coordinates. However, it is less obvious that there exist many more other cubes sitting in space as in Figure 1 (a). As a curiosity, our counting shows that there are precisely 242,483,634 cubes with vertices in $\{0, 1, \dots, 100\}^3$ and one non-trivial example of these cubes is given by the points

$$\{[0, 56, 59], [21, 68, 3], [24, 0, 56], [45, 12, 0], [52, 77, 83], [73, 89, 27], [76, 21, 80], [97, 33, 24]\} := \mathcal{C}.$$

In [13] we proved the following theorem.

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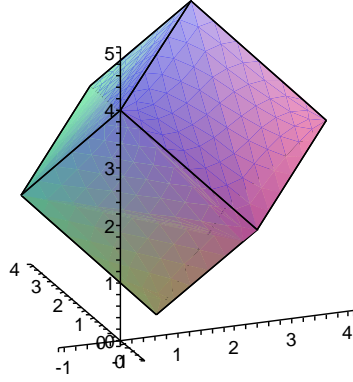
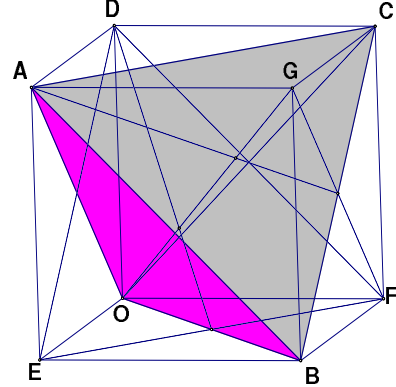


Figure 1(a): Non-trivial cube

Figure 1(b): Regular tetrahedron inscribed: $OABC$

THEOREM 1.1. *Every regular tetrahedron in \mathbb{Z}^3 can always be completed to a cube in \mathbb{Z}^3 (See Figure 1 (b)).*

This theorem implies that there is a one-to-two correspondence between the cubes and the regular tetrahedrons in \mathbb{Z}^3 . In [11] we have developed a Maple code to compute the number regular tetrahedrons in $\{0, 1, 2, \dots, n\}^3$. We will basically use the same idea and introduce some updates based on important theoretical observations. The problem of finding the number of cubes in space with coordinates in $\{0, 1, 2, \dots, n\}$ has been studied also in [14]. We list here a few more terms in the sequence A098928.

n	1	2	3	4	5	6	7	8	9	10	11
A098928	1	9	36	100	229	473	910	1648	2795	4469	6818

n	12	13	14	15	16	17	18
A098928	10032	14315	19907	27190	36502	48233	62803

It is clear that $A098928 \leq A103158$. For $n \geq 4$ we have actually a strict inequality, $A098928 < A103158$, and this is due to the fact that some of the tetrahedrons inside of the grid $\{0, 1, \dots, n\}^3$ extend beyond the grid's boundaries to the unique cube containing it. In Figure 2 we have included the two graphs of the sequences A098928 and A103158 up to $n = 100$.

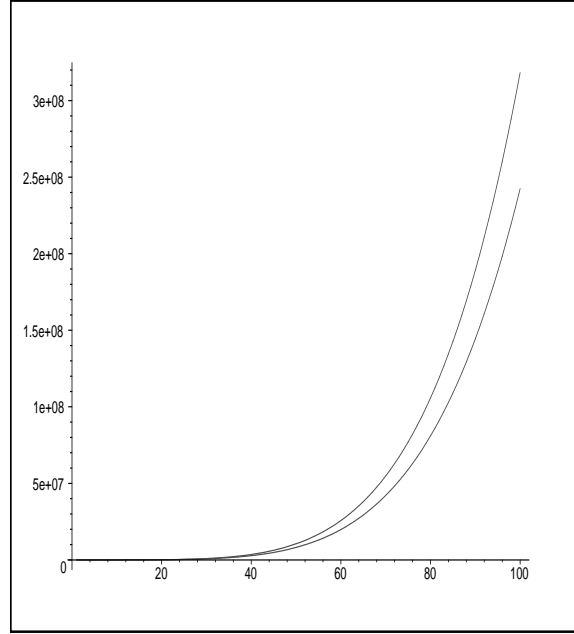


Figure 2: Tetrahedra versus cubes

2. THEORETICAL BACKGROUND

Let us review some of the facts that we are using. Regular tetrahedra are going to be obtained from equilateral triangles. Equilateral triangle in \mathbb{Z}^3 are obtained in the following way. Given an odd integer d there is a very precise number of solutions for the Diophantine equation (see [11]),

$$(1) \quad a^2 + b^2 + c^2 = 3d^2, \text{ with } 0 < a \leq b \leq c \text{ and } \gcd(a, b, c) = 1$$

which is given by

$$(2) \quad \pi\epsilon(d) = \frac{\Lambda(d) + 24\Gamma_2(d)}{48},$$

where

$$(3) \quad \Gamma_2(d) = \begin{cases} 0 & \text{if } d \text{ is divisible by a prime factor of the form } 8s+5 \text{ or } 8s+7, \ s \geq 0, \\ 1 & \text{if } d \text{ is } 3 \\ 2^k & \begin{cases} \text{where } k \text{ is the number of distinct prime factors of } d \\ \text{of } d \text{ of the form } 8s+1, \text{ or } 8s+3 \ (s > 0), \end{cases} \end{cases}$$

$$(4) \quad \Lambda(d) := 8d \prod_{p|d, p \text{ prime}} \left(1 - \frac{\left(\frac{-3}{p}\right)}{p}\right),$$

and $\left(\frac{-3}{p}\right)$ is the Legendre symbol. In particular, these precise countings show that the equation (1) has solutions for every odd $d \geq 1$.

We remind the reader that, if p is an odd prime then

$$(5) \quad \left(\frac{-3}{p}\right) = \begin{cases} 0 & \text{if } p = 3 \\ 1 & \text{if } p \equiv 1 \text{ or } 7 \pmod{12} \\ -1 & \text{if } p \equiv 5 \text{ or } 11 \pmod{12} \end{cases}.$$

As an example, for $d = 2011 = 251(8) + 3 = 167(12) + 7$, which is a prime, will give $\Lambda(d) = 16080$ and $\Gamma_2(d) = 48$, and so $\pi\epsilon(2011) = \frac{16080+48}{48} = 336$. There is only one solution, in this case, for which the three values of a , b , and c are not all distinct: $a = 139$ and $b = c = 2461$.

For each solution of (1) one can find a two integer parameter family of equilateral triangle in \mathbb{Z}^3 , with vertices in a plane through any point of integer coordinates (so we can take simply the origin O) and normal $\frac{(a,b,c)}{(d\sqrt{3})}$. Such an equilateral triangle, say $\triangle OPQ$, can be given terms of two vectors $\vec{\zeta}$ and $\vec{\eta}$ described by the next formulae.

$$(6) \quad \overrightarrow{OP} = m\vec{\zeta} - n\vec{\eta}, \quad \overrightarrow{OQ} = n\vec{\zeta} - (n-m)\vec{\eta}, \quad \text{with } \vec{\zeta} = (\zeta_1, \zeta_1, \zeta_2), \vec{\eta} = (\eta_1, \eta_2, \eta_3),$$

$$(7) \quad \begin{cases} \zeta_1 = -\frac{rac + dbs}{q}, \\ \zeta_2 = \frac{das - bcr}{q}, \\ \zeta_3 = r, \end{cases}, \quad \begin{cases} \eta_1 = -\frac{db(s - 3r) + ac(r + s)}{2q}, \\ \eta_2 = \frac{da(s - 3r) - bc(r + s)}{2q}, \\ \eta_3 = \frac{r + s}{2}, \end{cases}$$

where $q = a^2 + b^2$ and (r, s) is a suitable solution of $2q = s^2 + 3r^2$ that makes all the numbers in (7) integers. The sides-lengths of $\triangle OPQ$ are equal to $d\sqrt{2(m^2 - mn + n^2)}$.

One way to give a more precise construction of a good choice of (r, s) is this to compute the greatest common divisor, $s + i\sqrt{3}r$, of $A - i\sqrt{3}B$ and $2q$ in the ring $\mathbb{Z}[i\sqrt{3}]$, where $A = ac$ and $B = bd$.

Let us observe that $A^2 + 3B^3 = (ac)^2 + 3(bd)^2 = (a^2 + b^2)(c^2 + b^2)$ which shows that $2q$ divides $A^2 + 3B^2 = (A + i\sqrt{3}B)(A - i\sqrt{3}B)$. Since $2q = 4(4k + 1)$ for some integer k , we are thinking of 4 as $(1 + i\sqrt{3})(1 - i\sqrt{3})$, so the prime factors of $2q$ here are given by $1 + i\sqrt{3}$, $1 - i\sqrt{3}$ and all the others that appear which are either primes of the form $6k - 1$ or of the form $6k + 1$. The factors of the form $6k - 1$ must appear to even power and those of the form $6k + 1$ can be decomposed into prime factors $u + i\sqrt{3}v$ and $u - i\sqrt{3}v$ (by Euler's Theorem). Each of these factors can be found either in $A + i\sqrt{3}B$ or $A - i\sqrt{3}B$. The product of these factors give $s + i\sqrt{3}r$. By construction $A - i\sqrt{3}B = (s + i\sqrt{3}r)(u + i\sqrt{3}v)$ and $2q = (s + i\sqrt{3}r)(s - i\sqrt{3}r)$ because of properties of conjugation of complex numbers. This implies that $(A - i\sqrt{3}B)(s - i\sqrt{3}r) = 2q(u + i\sqrt{3}v)$.

Hence we get the relations

$$2q = s^2 + 3r^2, \quad As - 3Br = 2qu \quad \text{and} \quad Ar + Bs = -2qv.$$

These two relations show that ζ_1 and η_1 in (7) are integers. Lagrange's identity shows that

$$\zeta_1^2 + \zeta_2^2 = \frac{(a^2 + b^2)((rc)^2 + (ds)^2)}{q^2} = \frac{(rc)^2 + (ds)^2}{a^2 + b^2},$$

which in turn gives

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = \frac{r^2(a^2 + b^2) + r^2c^2 + d^2s^2}{a^2 + b^2} = \frac{d^2(s^2 + 3r^2)}{q} = 2d^2.$$

This implies in particular that ζ_2 must be an integer and that $|\vec{\zeta}| = d\sqrt{2}$.

It is clear that r and s must be either both odd or both even. This implies that η_3 is an integer. Using again Lagrange's identity we get

$$\eta_1^2 + \eta_2^2 = \frac{(a^2 + b^2)[c^2(r + s)^2 + d^2(s - 3r)^2]}{4q^2} = \frac{c^2(r + s)^2 + d^2(s - 3r)^2}{4(a^2 + b^2)},$$

which implies

$$\eta_1^2 + \eta_2^2 + \eta_3^2 = \frac{(r + s)^2(a^2 + b^2 + c^2) + d^2(s - 3r)^2}{4(a^2 + b^2)} = \frac{d^2[3(r + s)^2 + (s - 3r)^2]}{4q} = 2d^2.$$

As before, this proves that η_2 is an integer and $|\vec{\eta}| = d\sqrt{2}$.

In order to find the dot product of $\vec{\zeta}$, $\vec{\eta}$ we observe that

$$\zeta_1\eta_1 + \zeta_2\eta_2 = \frac{(a^2 + b^2)[c^2(r^2 + rs) + d^2(s^2 - 3rs)]}{2q^2}$$

which implies

$$\vec{\zeta} \cdot \vec{\eta} = \frac{c^2(r^2 + rs) + d^2(s^2 - 3rs)}{2q} + \frac{r^2 + rs}{2} = \frac{d^2(3r^2 + 3rs + s^2 - 3rs)}{2q} = d^2.$$

Hence the angle between the vectors $\vec{\zeta}$, $\vec{\eta}$ is $\arccos(\frac{\vec{\zeta} \cdot \vec{\eta}}{|\vec{\zeta}| |\vec{\eta}|}) = 60^\circ$.

Using these relations, we can easily calculate

$$|\vec{OP}|^2 = m^2 |\vec{\zeta}|^2 - 2 \vec{\zeta} \cdot \vec{\eta} mn + n^2 |\vec{\eta}|^2 = 2d^2(m^2 - mn + n^2), \text{ and}$$

$$|\vec{OQ}|^2 = n^2 |\vec{\zeta}|^2 - 2 \vec{\zeta} \cdot \vec{\eta} n(n-m) + (n-m)^2 |\vec{\eta}|^2 = 2d^2[n^2 - n(n-m) + (n-m)^2] = 2d^2(m^2 - mn + n^2).$$

The dot product of \vec{OP} and \vec{OQ} is then equal to

$$\vec{OP} \cdot \vec{OQ} = mn |\vec{\zeta}|^2 - [m(n-m) + n^2] \vec{\zeta} \cdot \vec{\eta} + n(n-m) |\vec{\eta}|^2 =$$

$$d^2(2mn - mn + m^2 - n^2 + 2n^2 - 2mn) = d^2(m^2 - mn + n^2).$$

These relations show that the triangle $\triangle OPQ$ is indeed equilateral and its side lengths are equal to $d\sqrt{m^2 - mn + n^2}$. One can easily check that

$$a\zeta_1 + b\zeta_2 + c\zeta_3 = a\zeta_1 + b\eta_2 + c\eta_3 = 0,$$

which implies that $\triangle OPQ$ is indeed contained in the plane of normal $\vec{n} = \frac{(a,b,c)}{d\sqrt{3}}$.

One good question here is, whether or not there are other equilateral triangles with integer coordinates contained in the same plane.

We have proved this in [2]. However, for completion we will include a new relatively simpler argument and with a more geometric flavor. If there exists one such triangle, say $\triangle OAB$, we may assume one of its vertices is at the origin (by implementing a translation with integer coordinates). Because the vectors $\vec{\zeta}$ and $\vec{\eta}$ form a basis for the space of vectors perpendicular to \vec{n} , the equation $\vec{OA} = m\vec{\zeta} - n\vec{\eta}$ can be solved uniquely for real numbers m and n . Let us consider then the vectors

$$\vec{OB'} = n\vec{\zeta} - (n-m)\vec{\eta}, \quad \vec{OB''} = (m-n)\vec{\zeta} - m\vec{\eta}.$$

With the same computations as before this will give two equilateral triangles in the same plane, namely $\triangle OAB'$ and $\triangle OAB''$. Because there are only two equilateral triangles sharing the side OA in the given plane, we must have either $B' = B$ or $B'' = B$. Without loss of generality, let us assume that $B' = B$. From the formulae in (7) we get that

$$mr - \frac{r+s}{2}n = u \in \mathbb{Z}, \quad \text{and} \quad nr - \frac{r+s}{2}(n-m) = \frac{r+s}{2}m + \frac{r-s}{2}n = v \in \mathbb{Z}.$$

If we look at these two equations a system of equations in m and n , we get by Cramer's formula, a unique solution which are going to be (rational numbers) fractions with integer numerator as the

denominators equal to

$$r \frac{r-s}{2} + \frac{r+s}{2} \frac{r+s}{2} = \frac{s^2 + 3r^2}{4} = \frac{q}{2} = \frac{a^2 + b^2}{2} \geq 1.$$

The same calculations as before show that the side length of $\triangle OAB$ is given by $\ell = d\sqrt{2(m^2 - mn + n^2)}$.

We can do this whole construction for b and c , or for a and c , instead of a and b . We get that

$$(8) \quad \ell^2 = 2d^2(m^2 - mn + n^2) = 2d^2(m_1^2 - m_1n_1 + n_1^2) = 2d^2(m_2^2 - m_2n_2 + n_2^2)$$

with m_1, n_1, m_2, n_2 rational numbers with $\frac{b^2+c^2}{2}$ and $\frac{a^2+c^2}{2}$ at the denominators, respectively. But the $\gcd(\frac{a^2+b^2}{2}, \frac{b^2+c^2}{2}, \frac{a^2+c^2}{2})$ cannot be greater than one since $\gcd(a, b, c) = 1$. Hence, in (8) we cannot have the number $m^2 - mn + n^2 = m_1^2 - m_1n_1 + n_1^2 = m_2^2 - m_2n_2 + n_2^2$ be a fraction in the reduced form with a denominator greater than one since any prime dividing it will divide $\gcd(\frac{a^2+b^2}{2}, \frac{b^2+c^2}{2}, \frac{a^2+c^2}{2}) = 1$. So, we proved that the triangle $\triangle OAB$ (or any other triangle with integer coordinates in the plane containing the origin of normal \vec{n}) has sides at least $d\sqrt{2}$.

Now, if m and n are not integers, then A and B fall strictly inside of the tessellation with equilateral triangles generated by the two vectors $\vec{\zeta}$ and $\vec{\eta}$ (see Figure 3). Because the tessellation is invariant to 60° the position of B inside one of the equilateral triangles is perfectly the same as the position of A inside of its equilateral triangle. This creates two vectors of the length and one is the rotation of the other by 60° . Using translations with integer coordinates the two vectors show the existence of a triangle with one vertex the origin, $\triangle OCD$, which is equilateral having integer coordinates and side lengths strictly less than $d\sqrt{2}$. This contradiction shows that A and B must be vertices of the tessellation generated by $\vec{\zeta}$ and $\vec{\eta}$, and so the parametrization (7) is unique.

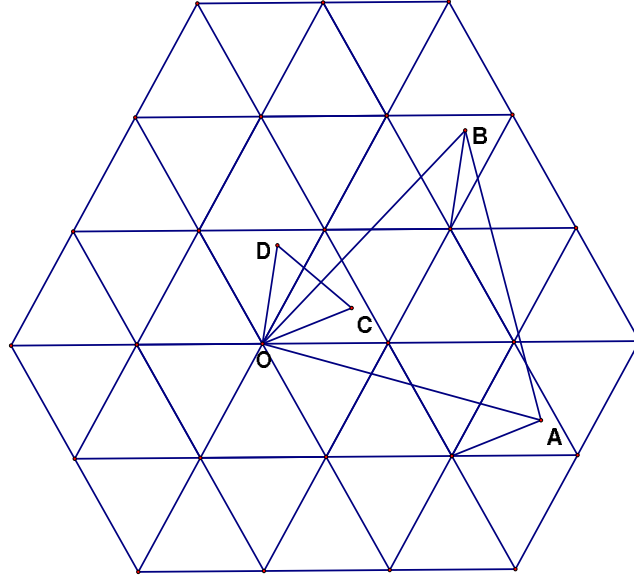


Figure 3: Two distinct tessellations

We have given then another proof of Theorem 1 in [10].

We are including here an example that is illustrating this parametrization. For $d = 2011$ we have seen a solution of (1): $a = 139$ and $b = c = 2461$. If we do the parametrization with $q = a^2 + b^2 = (2)(3037921)$ (3037921 is prime). Since $A = ac = (23)(107)(139)$ and $B = (23)(107)(2011)$, we get

$$A - iB = (23)(107)(1 - \sqrt{3}i)(1543 - 468\sqrt{3}i)$$

and

$$2q = (1 - \sqrt{3}i)(1543 - 468\sqrt{3}i)(1 + \sqrt{3}i)(1543 + 468\sqrt{3}i)$$

which gives $s + r\sqrt{3}i = (1 - \sqrt{3}i)(1543 - 468\sqrt{3}i) = 139 - 2011\sqrt{3}i$

This gives

$$\begin{cases} \zeta_1 = 0 \\ \zeta_2 = 2011 \\ \zeta_3 = -2011 \end{cases} \quad \text{and} \quad \begin{cases} \eta_1 = -2461 \\ \eta_2 = 1075 \\ \eta_3 = -936. \end{cases}$$

One can check that, in fact, in the case $b = c$ we may always take

$$(9) \quad \begin{cases} \zeta_1 = 0 \\ \zeta_2 = d \\ \zeta_3 = -d \end{cases} \quad \text{and} \quad \begin{cases} \eta_1 = -b \\ \eta_2 = \frac{a+d}{2} \\ \eta_3 = \frac{a-d}{2}. \end{cases}$$

Let us summarize all these facts that we have shown so far.

THEOREM 2.1. *For every solution of the Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ there exists essentially only one parametrization for the equilateral triangles with vertices in the plane of equation $ax + by + cz = 0$. Every equilateral triangle in \mathbb{Z}^3 is given by such a parametrization. Moreover, up to a translation, the vertices of such a triangle are given by (6) and (7), where r and s can be computed by finding the greatest common divisor, $s + i\sqrt{3}r$, of $A - i\sqrt{3}B$ and $2q$ in the ring $\mathbb{Z}[i\sqrt{3}]$, where $A = ac$ and $B = bd$. In the case $b = c$, the formulae (7) simplify to (9).*

In [12], we have shown that the only equilateral triangles, in \mathbb{Z}^3 , which can be completed to a regular tetrahedron in \mathbb{Z}^3 , are the ones (given as in (6) and (7)) for which $m^2 - mn + n^2 = k^2$ for some $k \in \mathbb{Z}$. More precisely, if k is divisible by 3 then one can accomplish this on either side of the plane containing the triangle and if k is not divisible by 3 then this can be done on only one side. By the way, this is saying in particular that, there are a lot more equilateral triangles than regular tetrahedrons in \mathbb{Z}^3 .

The coordinates for the fourth vertex, assuming the equilateral triangle's vertices are as in (6) and (7), are given by

$$(10) \quad \left(\frac{(2\zeta_1 - \eta_1)m}{\pm 2ak}, \frac{(2\zeta_2 - \eta_2)m}{\pm 2bk}, \frac{(2\zeta_3 - \eta_3)m}{\pm 2ck} \right).$$

As we already mentioned in Theorem 1.1, as long as the coordinates in (10) are integers then the tetrahedron can be completed to a cube in \mathbb{Z}^3 . We are using this formula mostly for $k = 1$ (let us choose $m = 1$ and $n = 0$) although there is a need for the general case for big values of d because, as pointed out in [11], there are irreducible regular tetrahedra which cannot be constructed from a face as above, by simply taking $k = 1$. However for small ℓ ($\ell < 5187 = 3(7)(13)(19)$) one can find a face of a given regular tetrahedron of sides equal to $\ell\sqrt{2}$ which has the corresponding k as in (10) equal to 1.

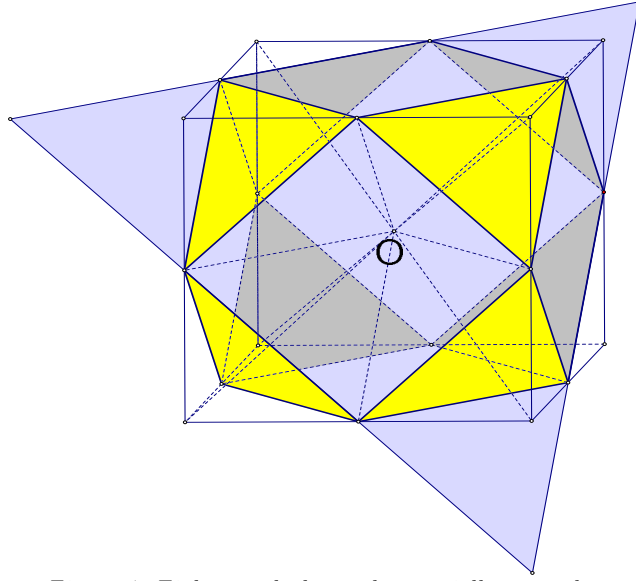


Figure 4: Eight tetrahedra and essentially one cube

If one takes all the possible values for m and n such that $m^2 - mn + n^2 = 1$ there are six regular tetrahedra generated this way, from a plane (colored blue in Figure 4), three on one side and the other three on the other side, but if one looks at the Figure 4, he/she might observe that in fact there are eight regular tetrahedra all generating essentially the same cube (up to translations of integer coordinates). As result, our code is going to reflect this property and we will only use one of the choices for the values for m and n . Summarizing, there are in general four planes containing the center of a given cube in \mathbb{Z}^3 , corresponding to normals given by the directions of the four big diagonals in the cube which may generate the the cube as before, some may have a value of $k > 1$. For this reason, one needs to check for repetitions when writing the code. For this purpose, our

approach is to generate an exhaustive list, \mathcal{L} , of cubes in \mathbb{N}_0^3 ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) which are irreducible (cannot be integer dilated to a smaller cube in \mathbb{Z}^3). One other property of each cube in \mathcal{L} is that it cannot be translated in the negative direction along any of the axes of coordinates and remain in \mathbb{N}_0^3 . However, the cubes in \mathcal{L} are not uniquely defined this way, because of the possible symmetries involved here. These symmetries are, in general, 48 in number and form a group which can be identified with the symmetry group of a regular octahedron (see [13]).

3. THE MINIMAL LIST AND OTHER CONSIDERATIONS

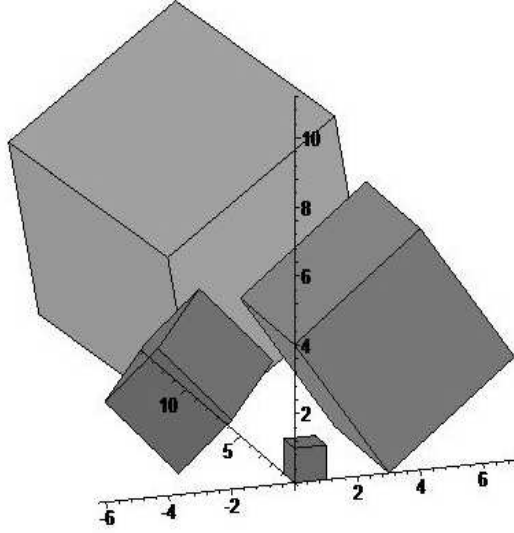


Figure 5: First cubes in \mathcal{L}

A dozen cubes (listed in non-decreasing order of their side-lengths) in the list \mathcal{L} that we found using our code, are included in the table below. The first column represents the side-lengths, the second column gives the dimension of the smallest cube $C_m := [0, m]^3$ containing the one in column three.

Table 1

n	m	A cube	k-values	invariants
1	1	[0, 0, 0], [0, 0, 1], [0, 1, 0], [0, 1, 1], [1, 0, 0], [1, 0, 1], [1, 1, 0], [1, 1, 1]	1	[1, 1, 0, 0]
3	5	[0, 3, 2], [1, 1, 4], [2, 2, 0], [2, 5, 3], [3, 0, 2], [3, 3, 5], [4, 4, 1], [5, 2, 3]	1,3	[4, 4, 0, 0]
5	7	[0, 0, 4], [0, 5, 4], [3, 0, 0], [3, 5, 0], [4, 0, 7], [4, 5, 7], [7, 0, 3], [7, 5, 3]	1	[12, 18, 4, 0]
7	11	[0, 6, 8], [2, 9, 2], [3, 0, 6], [5, 3, 0], [6, 8, 11], [8, 11, 5], [9, 2, 9], [11, 5, 3]	1,7	[8, 8, 0, 0]
9	15	[0, 5, 5], [4, 4, 13], [4, 13, 4], [7, 1, 1], [8, 12, 12], [11, 0, 9], [11, 9, 0], [15, 8, 8]	1,3	[24, 108, 48, 16]
11	19	[0, 11, 13], [2, 2, 7], [6, 17, 6], [8, 8, 0], [9, 9, 19], [11, 0, 13], [15, 15, 12], [17, 6, 6]	1	[24, 108, 48, 16]
13	19	[0, 12, 15], [3, 16, 3], [4, 0, 12], [7, 4, 0], [12, 15, 19], [15, 19, 7], [16, 3, 16], [19, 7, 4]	1, 13	[8, 8, 0, 0]
13	17	[0, 0, 12], [0, 13, 12], [5, 0, 0], [5, 13, 0], [12, 0, 17], [12, 13, 17], [17, 0, 5], [17, 13, 5]	1	[12, 30, 8, 0]
15	25	[0, 5, 10], [2, 19, 15], [10, 0, 20], [11, 7, 0], [12, 14, 25], [13, 21, 5], [21, 2, 10], [23, 16, 15]	1, 3	[48, 360, 176, 64]
17	29	[0, 20, 9], [1, 8, 21], [12, 12, 0], [12, 29, 17], [13, 0, 12], [13, 17, 29], [24, 21, 8], [25, 9, 20]	1	[24, 60, 16, 0]
17	23	[0, 0, 15], [0, 17, 15], [8, 0, 0], [8, 17, 0], [15, 0, 23], [15, 17, 23], [23, 0, 8], [23, 17, 8]	1	[12, 42, 12, 0]
19	31	[0, 16, 10], [6, 6, 25], [10, 31, 16], [15, 10, 0], [16, 21, 31], [21, 0, 15], [25, 25, 6], [31, 15, 21]	1, 19	[8, 8, 0, 0]

In the column four we list the values of k which can be used in the construction described in Section 2 to generate the cube in column three. The list of invariants are as follows. First, is the number

of cubes in the orbit obtained by applying the group of 48 transformations determined by the orthogonal matrices with coefficients 0 and 1. Let us denote this number by α_0 . Notice that this is a divisor of 48 as expected (Lagrange's Theorem). We expect that in general such a cube will have no special symmetry and so, more often we will get $\alpha_0 = 48$. The second number in the invariants list is the number of cubes in the *generalized* orbit, obtained by the previous orbit together with all its integer translations along the axes of coordinates that remain in C_m , a number that we are going to denote by α . The third number in the list, β , is the cardinality of the intersection between this former orbit and its translation along $(0, 0, 1)$. Finally, the last number, γ , is defined by the cardinality of the generalized orbit with its translation along $(0, -1, 1)$. It turns out that these last three numbers are enough to determine the number of cubes that one can fit by translating the given cube in all possible ways within a bigger cube of size $k \geq m$. This fact has been essentially proved in Theorem 2.2 in [10]. The formula that gives this number is

$$(11) \quad (k - m + 1)^3 \alpha - 3(k - m)(k - m + 1)^2 \beta + 3(k - m + 1)(k - m)^2 \gamma.$$

One of the observations that we will make, about Table 1, is that this set of invariants is not complete, since we see that the same numbers appear for various irreducible cubes. The most surprising are those cubes in rows six and seven. A good problem here is to determine the exact number of such cubes, which go into a certain side-length n , in terms of n . We see that the first n for which we have two such cubes is $n = 13$. Let us also observe that in column four we see a 1 in there for each cube. As we mentioned earlier, this is not always the case.

Also, each cube in Table 1 with side-lengths n , gives rise to an orthogonal matrix with rational coefficients having denominators in $\frac{1}{n}\mathbb{Z}$ (obtained by taking the normalized vectors along the sides of the cube that form an orthogonal basis). In [13] we computed a few of them which are included here next:

$$T_3 := \frac{1}{3} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{bmatrix}, T_5 := \frac{1}{5} \begin{bmatrix} 4 & 0 & 3 \\ 3 & 0 & -4 \\ 0 & -5 & 0 \end{bmatrix}, T_7 := \frac{1}{7} \begin{bmatrix} -2 & -3 & 6 \\ 3 & -6 & -2 \\ -6 & -2 & -3 \end{bmatrix}, T_9 := \frac{1}{9} \begin{bmatrix} -7 & -4 & 2 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{bmatrix}$$

$$T_{11} := \frac{1}{11} \begin{bmatrix} 2 & -9 & -6 \\ 9 & -2 & 6 \\ -6 & -6 & 7 \end{bmatrix}, T_{13} := \frac{1}{13} \begin{bmatrix} -4 & -12 & -3 \\ 12 & -3 & -4 \\ 3 & -4 & 12 \end{bmatrix}, \hat{T}_{13} := \frac{1}{13} \begin{bmatrix} 0 & -13 & 0 \\ 12 & 0 & 5 \\ -5 & 0 & 12 \end{bmatrix}.$$

The next matrix can be obtained by multiplying T_3 with T_5 . We notice a multiplicative structure on this set of matrices. For the next two prime sizes we have essentially two orthogonal matrices.

$$T_{17} := \frac{1}{17} \begin{bmatrix} 12 & -8 & -9 \\ 12 & 9 & 8 \\ 1 & -12 & 12 \end{bmatrix}, \hat{T}_{17} := \frac{1}{17} \begin{bmatrix} 15 & 0 & 8 \\ 8 & 0 & -15 \\ 0 & -17 & 0 \end{bmatrix},$$

$$T_{19} := \frac{1}{19} \begin{bmatrix} 6 & -18 & 1 \\ 17 & 6 & 6 \\ -6 & -1 & 18 \end{bmatrix}, \hat{T}_{19} := \frac{1}{19} \begin{bmatrix} 15 & -6 & -10 \\ 10 & 15 & 6 \\ 6 & -10 & 15 \end{bmatrix}.$$

From T_5 , \hat{T}_{13} , and \hat{T}_{19} it is clear that there is a natural imbedding of the primitive Pythagorean Triples into this sequence of orthogonal matrices (well known in the literature). One interesting question here is the following: what is the algebraic relevance for the geometric invariants α_0 , α , β and γ , for an orthogonal matrix as above? One more observation here: each cube in the list generates by translations and rotations cubes in C_k and two different cubes cannot generate the same cube because by doing those transformations the four planes given by the diagonals are preserved and those are different for two different cubes in the list. So, in order to count all the cubes in C_k , we first compute the list of irreducible cubes in \mathcal{L} , up to the side length k , and then for each one we use the formula (11) to find out how many are generated by each in C_k . This is not enough though because there are cubes in C_k which are not irreducible. So, in the end we multiply each cube in the list \mathcal{L} by an integer factor in such a way the resulting cube can still fit in C_k . Then, we recalculate the invariants on this cube and use the same formula (11) to find the contribution of the reducible cubes. In the end, we add up all these numbers and that gives, $NC(k)$, the number of cubes in C_k .

The first 100 values of NC are: 1, 9, 36, 100, 229, 473, 910, 1648, 2795, 4469, 6818, 10032, 14315, 19907, 27190, 36502, 48233, 62803, 80736, 102550, 128847, 160271, 197516, 241314, 292737, 352591, 421764, 501204, 592257, 696281, 814450, 948112, 1098607, 1267367, 1456292, 1666998, 1901633, 2162179, 2450440, 2768346, 3117935, 3501389, 3923178, 4384792, 4889323, 5439155, 6037660, 6687358, 7391669, 8154671, 8979750, 9870158, 10830095, 11862711, 12972046, 14161848, 15436931, 16801993, 18263634, 19825948, 21493019, 23269647, 25160816, 27171482, 29308957, 31577319, 33986616, 36540004, 39244371, 42106267, 45131996, 48327502, 51700279, 55258019, 59011634, 62965766, 67132037, 71515527, 76127374, 80973598, 86062187, 91401297, 96999986, 102866282, 109014085, 115457359, 122206348, 129266410, 136648555, 144364071, 152426724, 160843660, 169626467, 178787563, 188347314, 198309846, 208694461, 219509943, 230767760, and 242483634.

4. THE CODE

The code is written in Maple and it is attached in pdf format at the end of the paper after bibliography.

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```
> restart:with(numtheory):with(plots):
```

1 (Step 1) Procedures to find the possible values of k, m and n, and normal plane vector (a,b,c)

Find all values of k less than n such that k is of the form a product of primes of the form $3k+1$

```
> kvalues:=proc(n)
> local i,j,k,L,a,p,q,r,m,mm;
> L:={};mm:=floor((n+1)/2);
> for i from 2 to mm do
> a:=ifactors(2*i-1);
> k:=nops(a[2]);r:=0;
> for j from 1 to k do
> m:=a[2][j][1]; p:=m mod 3;
> if m=3 then r:=1 fi;
> if r=0 and p=2 then r:=1 fi;
> od;
> if r=0 then L:=L union {2*i-1};fi;
> od;
> L:=L union {1};
> L:=convert(L,list);
> end:
```

Find solutions (m,n) of the equation $k^2=m^2-mn+n^2$, $\gcd(m,n)=1$, $m>0$, $n>0$, $2m<n$.

```
> listofmn:=proc(k)
> local a,b,i,nx,x,m,n,L,b1,b2;
> x:=[isolve(k^2=m^2-m*n+n^2)];
> nx:=nops(x);
> L:={};
> for i from 1 to nx do
> if lhs(x[i][1])=m then a:=rhs(x[i][1]); b:=rhs(x[i][2]);
> else b:=rhs(x[i][1]); a:=rhs(x[i][2]); fi;
> if gcd(a,b)=1 and a>=0 and b>0 and 2*a<b then L:=L union
> {[a,b]};fi;
> od;L;
> end:
```

Find Only the solutions that satisfy $\gcd(a,b,c)=1$ and $0<a\leq b\leq c$ are in the input.

```

> abcsol:=proc(d)
> local i,j,k,m,u,x,y,sol,cd;sol:={};
> for i from 1 to d do
> u:=[isolve(3*d^2-i^2=x^2+y^2)];k:=nops(u);
> for j from 1 to k do
> if rhs(u[j][1])>=i and rhs(u[j][2])>=i then
> cd:=gcd(gcd(i,rhs(u[j][1])),rhs(u[j][2]));
> if cd=1 then sol:=sol union
> {sort([i,rhs(u[j][1]),rhs(u[j][2]))];fi;
> fi;
> od;
> od;
> convert(sol,list);
> end:

```

2 (Step2) The new method of finding r and s.
It is followed by the implementation of the parametrization of equilateral triangles constructing only one regular tetrahedron and completing it to a cube (the origin is one of its vertices)

Procedure to get the unique writing as x^2+3y^2 for a prime p of the form $6k+1$.

```

> uniquedecomposition:=proc(p)
> local out,s,out1,out2;
> if p=2 then out:=[1,1]; fi;
> if p>2 then s:=isolve(p=x^2+3*y^2);
> out1:=abs(rhs(s[1][1]));out2:=abs(rhs(s[1][2]));
> if out1^2+3*out2^2=p then out:=[out1,out2]; else
> out:=[out2,out1];fi;
> fi;
> out; end:

```

The next procedure is calculating the factorization in $\mathbb{Z}[\sqrt{3}i]$ of a number of the form $u+v\sqrt{3}i$*

```

> factoroverEisensteinintegers:=proc(u,v)
> local i,N,M,k,a,x,f1,f2,g,y,y1,y2,L,NN,MM,uu,vv;
> a:=sqrt(3)*I;
> NN:=gcd(u,v);uu:=u/NN;vv:=v/NN;
> N:=uu^2+3*vv^2;
> x:=uu+vv*a;
> M:=ifactors(N);k:=nops(M[2]);
> for i from 1 to k do
> f1:=M[2][i][1]; f2:=M[2][i][2];
> if f1>2 then
> g:=uniquedecomposition(f1);
> else g:=[1,1]; f2:=1;
> fi;
> y:=expand(rationalize(x/(g[1]+a*g[2])));
> y1:=Re(y);
> y2:=type(y1,integer);
> if y2=true then L[i]:=[g[1]+a*g[2],f2]; else
> L[i]:=[g[1]-a*g[2],f2];
> fi;
> od;
> [NN,seq(L[i],i=1..k),expand(NN*product(L[ii][1]^L[ii][2],ii=1..k))];
> end:

```

Finding the gcd between $A+isqrt(3)B$ and $2(q)$

```

> findgcd:=proc(A,B,q)
> local
> i,j,f,common,m,qq,fac,a,L,Important,nfac,f1,ff1,f2,s,LL,n,P,rs,LLL;
> a:=sqrt(3)*I;L:={};
> f:=factoroverEisensteinintegers(A,B);m:=nops(f)-1;
> common:=gcd(f[1],2*q);
> qq:=2*q/common^2;
> P:=common;
> fac:=ifactors(qq);nfac:=nops(fac[2]);
> for i from 1 to nfac do
> f1:=fac[2][i][1];f2:=fac[2][i][2];
> if f1=2 then f2:=1;fi;
> s:=uniquedecomposition(f1);
> for j from 1 to m do
> if s[1]+a*s[2]=f[j][1] then
> P:=P*(s[1]+a*s[2])^(min(f[j][2],f2)); fi;
> if s[1]-a*s[2]=f[j][1] then
> P:=P*(s[1]-a*s[2])^(min(f[j][2],f2)); fi;
> od;
> od;
> P:=expand(P);
> rs:=[Re(P),Im(P)/sqrt(3)];
> [rs,A*rs[1]+3*B*rs[2] mod 2*q,A*rs[2]-B*rs[1] mod 2*q]
> end:

```

The fourth point to get a regular tetrahedron is the origin. We have included only one since all the other are going to produce the same cube when transformed into the positive quadrant.


```

> findpar:=proc(a,b,c,mm,nn)
> local i,j,sol,mx,nx,r,s,my,ny,q,d,u,v,w,x,y,z,mu,nu,mv,nv,ef,ns,mz,nz
> ,mw,nw,T,l,uu,t,R1,R2,fc,k,A,B,rs,G,DD,E,F;
> q:=a^2+b^2;k:=sqrt(mm^2-mm*nn+nn^2);d:=sqrt((a^2+b^2+c^2)/3);A:=a*c;B:
> =b*d;
> rs:=findgcd(A,-B,q);
> r:=rs[1][2]; s:=rs[1][1];
> uu:=(s^2+3*r^2-2*q)^2; if uu>0 then t:=s;s:=r;r:=t; fi;
> mx:=(d*b*(3*r+s)+a*c*(r-s))/(2*q);nx:=(r*a*c+d*b*s)/q;
> my:=(d*a*(3*r+s)-b*c*(r-s))/(2*q);ny:=(r*b*c-d*a*s)/q;
> mz:=(r-s)/2;nz:=r;
> mu:=nx;mv:=ny;mw:=nz;nu:=nx-mx;nv:=ny-my; nw:=nz-mz;
> u:=mu*m-nu*n;v:=mv*m-nv*n;w:=mw*m-nw*n;
> x:=mx*m-nx*n;y:=my*m-ny*n;z:=mz*m-nz*n;
> R2:=[(x+u-2*a*k)/3,(v+y-2*b*k)/3,(z+w-2*c*k)/3];
> R1:=[(x+u+2*a*k)/3,(v+y+2*b*k)/3,(z+w+2*c*k)/3];
> fc:=subs(m=mm,n=nn,R1[1]);
> if fc=floor(fc) then
> T:=subs(m=mm,n=nn,[[u,v,w],[x,y,z],R1]); else
> T:=subs(m=mm,n=nn,[[u,v,w],[x,y,z],R2]);
> fi;
> G:=[(T[1][1]+T[2][1]+T[3][1])/2,(T[1][2]+T[2][2]+T[3][2])/2,(T[1][3]+
> T[2][3]+T[3][3])/2];
> DD:=[(T[1][1]+T[2][1]-T[3][1])/2,(T[1][2]+T[2][2]-T[3][2])/2,(T[1][3]
> +T[2][3]-T[3][3])/2];
> E:=[(T[1][1]-T[2][1]+T[3][1])/2,(T[1][2]-T[2][2]+T[3][2])/2,(T[1][3]-
> T[2][3]+T[3][3])/2];
> F:=[(-T[1][1]+T[2][1]+T[3][1])/2,(-T[1][2]+T[2][2]+T[3][2])/2,(-T[1][
> 3]+T[2][3]+T[3][3])/2];
> [T[1],T[2],T[3],G,DD,E,F];
> end:

```

3 (Step 3) Procedures for adding vectors, other operations on vertices, finding the four k-values of a cube (most of the time is less than four)

```

> subtrv:=proc(U,V)
> local W;
> W[1]:=U[1]-V[1];W[2]:=U[2]-V[2];W[3]:=U[3]-V[3];
> [W[1],W[2],W[3]];
> end:
> addvec:=proc(U,V)
> local W;
> W[1]:=U[1]+V[1];W[2]:=U[2]+V[2];W[3]:=U[3]+V[3];
> [W[1],W[2],W[3]];
> end:

```

```

> addvectCube:=proc(T,v)
> local i,Q;Q:={};
> for i from 1 to 8 do
> Q:=Q union {addvec(T[i],v)};
> od;
> Q;
> end:
> distance:=proc(A,B)
> local C;
> C:=subtrv(A,B);
> sqrt(C[1]^2+C[2]^2+C[3]^2);
> end:
> finddiagonal1:=proc(C)
> local i,m,L,x,y,CC;
> L:=convert(C,list);m:=nops(L);
> for i from 1 to m do
> x:=type(distance(L[1],L[i])/sqrt(3),integer);
> if x=true then y:=[L[1],L[i]];fi;
> od;
> CC:=C minus {y[1],y[2]};
> y,CC;
> end:
> finddiagonal:=proc(C)
> local x,x1,x2;
> x:=finddiagonal1(C);
> x1:=finddiagonal1(x[2]);
> x2:=finddiagonal1(x1[2]);
> [x[1],x1[1],x2[1],convert(x2[2],list)];
> end:
> unitvector:=proc(U)
> local i,j,k,l,x;
> i:=U[1];j:=U[2];k:=U[3];
> l:=gcd(gcd(i,j),k);
> x:=(i^2+j^2+k^2)/(3*l^2);
> sqrt(x);
> end:
> fourkvalues:=proc(C)
> local N1,N2,N3,N4,x,length;length:=distance(C[1],C[2]);
> x:=finddiagonal(C);
> N1:=unitvector(subtrv(x[1][1],x[1][2]));
> N2:=unitvector(subtrv(x[2][1],x[2][2]));
> N3:=unitvector(subtrv(x[3][1],x[3][2]));
> N4:=unitvector(subtrv(x[4][1],x[4][2]));
> {length/N1,length/N2,length/N3,length/N4};
> end:
> multbyfactorv:=proc(v,k)
> local w;
> w[1]:=v[1]*k;w[2]:=v[2]*k;w[3]:=v[3]*k;
> [w[1],w[2],w[3]];
> end:

```

```

> multbyfactor:=proc(T,k)
> local i,NT,Q;NT:={};
> Q:=convert(T,list);
> for i from 1 to 8 do
> NT:=NT union {multbyfactorv(Q[i],k)};
> od;NT;
> end:

```

- 4 (Step 4) Translation of the cube obtained in Step 2 into the first quadrant and getting the orbit and the generalized orbit, then intersections with its translations along $(0,0,1)$ and $(0,1,-1)$. Calculation of α_0 , α , β and γ for a cube.

Translating the cube to the first octant ($x \geq 0, y \geq 0, z \geq 0$)

```

> tmttopcube:=proc(T)
> local i,a,b,c,v,O,C;
> a:=min(T[1][1],T[2][1],T[3][1],T[4][1],T[5][1],T[6][1],T[7][1],0);
> b:=min(T[1][2],T[2][2],T[3][2],T[4][2],T[5][2],T[6][2],T[7][2],0);
> c:=min(T[1][3],T[2][3],T[3][3],T[4][3],T[5][3],T[6][3],T[7][3],0);
> O:=[0,0,0];
> v:=[a,b,c];C:=
> {subtrv(O,v),subtrv(T[1],v),subtrv(T[2],v),subtrv(T[3],v),
> subtrv(T[4],v),subtrv(T[5],v),subtrv(T[6],v),subtrv(T[7],v)};
> end:
> mscofmcube:=proc(Q)
> local a,b,c,T;
> T:=convert(Q,list);
> a:=max(T[1][1],T[2][1],T[3][1],T[4][1],T[5][1],T[6][1],T[7][1],T[8][1]
> );
> b:=max(T[1][2],T[2][2],T[3][2],T[4][2],T[5][2],T[6][2],T[7][2],T[8][2]
> );
> c:=max(T[1][3],T[2][3],T[3][3],T[4][3],T[5][3],T[6][3],T[7][3],T[8][3]
> );
> max(a,b,c);
> end:

```

```

> orbit1Cube:=proc(T)
> local
> i,k,T1,a,b,c,x,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,T15,T16,T17
> ,T18,
> T19,T20,T21,T22,T23,T24,S,Q,d,a1,b1,c1;
> Q:=convert(T,list);
> d:=mscofmcube(T);
> T1:=T;
> T2:={seq([Q[k][2],Q[k][3],Q[k][1]],k=1..8)};
> T3:={seq([Q[k][1],Q[k][3],Q[k][2]],k=1..8)};
> T4:={seq([Q[k][1],Q[k][2],d-Q[k][3]],k=1..8)};
> T5:={seq([Q[k][2],Q[k][3],d-Q[k][1]],k=1..8)};
> T6:={seq([Q[k][1],Q[k][3],d-Q[k][2]],k=1..8)};
> T7:={seq([Q[k][1],d-Q[k][2],Q[k][3]],k=1..8)};
> T8:={seq([Q[k][2],d-Q[k][3],Q[k][1]],k=1..8)};
> T9:={seq([Q[k][1],d-Q[k][3],Q[k][2]],k=1..8)};
> T10:={seq([d-Q[k][1],Q[k][2],Q[k][3]],k=1..8)};
> T11:={seq([d-Q[k][2],Q[k][3],Q[k][1]],k=1..8)};
> T12:={seq([d-Q[k][1],Q[k][3],Q[k][2]],k=1..8)};
> T13:={seq([Q[k][1],d-Q[k][2],d-Q[k][3]],k=1..8)};
> T14:={seq([Q[k][2],d-Q[k][3],d-Q[k][1]],k=1..8)};
> T15:={seq([Q[k][1],d-Q[k][3],d-Q[k][2]],k=1..8)};
> T16:={seq([d-Q[k][1],d-Q[k][2],Q[k][3]],k=1..8)};
> T17:={seq([d-Q[k][2],d-Q[k][3],Q[k][1]],k=1..8)};
> T18:={seq([d-Q[k][1],d-Q[k][3],Q[k][2]],k=1..8)};
> T19:={seq([d-Q[k][1],Q[k][2],d-Q[k][3]],k=1..8)};
> T20:={seq([d-Q[k][2],Q[k][3],d-Q[k][1]],k=1..8)};
> T21:={seq([d-Q[k][1],Q[k][3],d-Q[k][2]],k=1..8)};
> T22:={seq([d-Q[k][1],d-Q[k][2],d-Q[k][3]],k=1..8)};
> T23:={seq([d-Q[k][2],d-Q[k][3],d-Q[k][1]],k=1..8)};
> T24:={seq([d-Q[k][1],d-Q[k][3],d-Q[k][2]],k=1..8)};
> S:=
> {T1,T2,T3,T4,T5,T6,T7,T8,T9,T10,T11,T12,T13,T14,T15,T16,T17,T18,T19,T
> 20,T21,T22,T23,T24};
> S;
> end:
> orbitCube:=proc(T)
> local S,Q,T1;
> Q:=convert(T,list);
> T1:={seq([Q[k][3],Q[k][2],Q[k][1]],k=1..8)};
> S:=orbit1Cube(T) union orbit1Cube(T1);
> S;
> end:

```

```

> translCube:=proc(T)
> local S,Q,i,j,k,a2,b2,c2,a,b,c,d,alpha;
> Q:=convert(T,list);
> a:=max(seq(Q[k][1],k=1..8));
> b:=max(seq(Q[k][2],k=1..8));
> c:=max(seq(Q[k][3],k=1..8));
> d:=max(a,b,c);
> a2:=d-a;b2:=d-b;c2:=d-c;
> S:=orbitCube(T);alpha:=nops(S);
> for i from 0 to a2 do
> for j from 0 to b2 do
> for k from 0 to c2 do
> S:=S union orbitCube(addvectCube(T,[i,j,k]));
> od;
> od;
> od;
> #alpha,nops(S),[a2,b2,c2],(a2*b2+b2*c2+c2*a2)*alpha;
> S;
> end:
> alphabetagamma:=proc(T)
> local x,S,i,S1,S2,d,cube,j,exitv,alpha,beta,gamma;
> d:=mscofmcube(T);x:=nops(orbitCube(T));
> S1:=translCube(T);S2:=S1;S:=convert(S2,list);alpha:=nops(S);
> for i from 1 to alpha do
> cube:=S[i];
> j:=1; exitv:=0;
> while j<=8 and exitv=0 do
> if cube[j][3]=d then exitv:=1;
> S2:=S2 minus {cube};fi;
> j:=j+1;
> od;
> od;
> beta:=nops(S2);S:=convert(S2,list);
> for i from 1 to beta do
> cube:=S[i];
> j:=1; exitv:=0;
> while j<=8 and exitv=0 do
> if cube[j][2]=d then exitv:=1;
> S2:=S2 minus {cube};fi;
> j:=j+1;
> od;
> od;
> gamma:=nops(S2);
> [S2,[x,alpha,beta,gamma]];
> end:

```

- 5 (Step 5) The function that gives the number of cubes generated by a small cube which fits in a cube of dimension d , in an arbitrary bigger cube of dimension n .

```
> f:=(n,d,alpha,beta,gamma)->(n-d+1)^3*alpha-3*(n-d)*(-d+1+n)^2*beta+3*
> gamma*(n-d+1)*(n-d)^2:
```

- 6 (Step 6) List of minimal (irreducible) cubes together with n , minimal size containing the cube, the k -values, α , β , and γ . In the end another list is created in a new procedure to get the list of the reducible cubes together with the α , β , and γ .

We compute the list of minimal cubes taking into account that at some point only $k=1$ is not sufficient to generate a cube .

```
> ExtendList:=proc(n,N,L::list,mm,nn,Orb::array)
> local i,sol,nsol,nel,ttcube,orb::array,
> LL::list,tnel,NL::list,cio,C,pC,k,kvalues,m,exception,abg,NN;
> NN:=floor((N-1)/2);
> orb:=array(1..2*NN+1);
> nel:=nops(L);
> LL:=L;
> k:=sqrt(mm^2-mm*nn+nn^2);
> m:=n*k;
> if m<=N then
> sol:=abcsol(n);nsol:=nops(sol);
> tnel:=nel;orb:=Orb;
> for i from 1 to nsol do
> C:=findpar(sol[i][1],sol[i][2],sol[i][3],mm,nn);pC:=tmtpqcube(C);
> kvalues:=fourkvalues(pC);exception:=evalb(1 in kvalues);
> if k=1 or exception=false then
> cio:=evalb(pC in orb[m]);
> if cio=false then
> abg:=alphabetagamma(pC);
> orb[m]:=orb[m] union abg[1];
> NL[tnel+1]:=[n,mscofmcube(pC),pC,kvalues,abg[2],sol[i]];
> tnel:=tnel+1;
> fi;
> fi;
> od;
> LL:=[seq(L[i],i=1..nel),seq(NL[j],j=tnel+1..tnel)];
> fi;
> LL,orb;
> end:
```

```

> ExtendListuptoN:=proc(N)
> local i,j,k,l,kv,kvn,NN,L,Orb::array,mn,nmn,E,n,ii;
> kv:=kvalues(N);kvn:=nops(kv);
> NN:=floor((N-1)/2);
> L:=[];Orb:=array(1..2*NN+1);
> for ii from 1 to 2*NN+1 do
>   Orb[ii]:={};
>   od;
>   for i from 1 to kvn do
>     k:=kv[i];mn:=listofmn(k);nmn:=nops(mn);
>     for j from 1 to nmn do
>       for l from 1 to NN+1 do
>         n:=2*l-1;
>         E:=ExtendList(n,N,L,mn[j][1],mn[j][2],Orb);
>         L:=E[1];
>         for ii from 1 to 2*NN+1 do
>           Orb[ii]:=E[2][ii];
>         od;
>       od;
>     od;
>   od;
>   L;
> end:

> L:=ExtendListuptoN(50):
> ExtendListuptoNmultiples:=proc(N,L)
> local i,j,x,lc,m,mm,d,dd,C,CC,LL;
> m:=nops(L);i:=1;LL:={};
> while i<=m do
>   d:=L[i][2];
>   if d<=N then
>     mm:=floor(N/d);C:=L[i][3];j:=2;
>     while j<=mm do
>       CC:=multbyfactor(C,j);x:=alphabetagamma(CC);
>       dd:=d*j;lc:=nops(LL);
>       LL:=LL union {[L[i][1]*j,dd,CC,L[i][4],x[2]]};
>       j:=j+1;
>     od;
>   fi;
>   i:=i+1;
> od;
> convert(LL,list);
> end:

> LL:=ExtendListuptoNmultiples(50,L):

```

7 (Step 7) Adding up the contribution of each cube.

```

> addupnew:=proc(N,L,LL)
> local i,j,k,nc,mm,m,d,C,CC,x,dd,nt;
> nc:=0;
> m:=nops(L);
> i:=1;
> while i<=m do
> d:=L[i][2];
> if d<=N then
> nc:=nc+f(N,d,L[i][5][2],L[i][5][3],L[i][5][4]);
> fi; i:=i+1;
> od;
> m:=nops(LL);
> i:=1;
> while i<=m do
> d:=LL[i][2];
> if d<=N then
> nc:=nc+f(N,d,LL[i][5][2],LL[i][5][3],LL[i][5][4]);
> fi; i:=i+1;
> od;
> nc;
> end:
> NC:=[seq(addupnew(k,L,LL),k=1..50)];

```

$NC := [1, 9, 36, 100, 229, 473, 910, 1648, 2795, 4469, 6818, 10032, 14315, 19907,$
 $27298, 36886, 49133, 64531, 83784, 107542, 136551, 171599, 213524, 263202,$
 $321849, 390415, 469932, 561492, 667305, 789317, 929098, 1088500, 1269367,$
 $1473635, 1703708, 1961706, 2251289, 2575291, 2936272, 3337026, 3780455,$
 $4269605, 4813854, 5414560, 6076915, 6804587, 7603120, 8476390, 9430481,$
 $10471175]$